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MIXTURES OF POISSON DISTRIBUTION

by

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BIOGRAPHICAL SKETCH

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ABSTRACT

A considerable volume of recent research work has been devoted to mixtures of discrete distributions. The purpose of this thesis is to obtain estimates of the parameters of the mixed-Poisson distribution and also to study their mathematical properties.

In Chapter 1, some recent work in these fields is reviewed, namely the work of Gumbel [15], Rider [26] and Blischke [4]. Chapter II, contains a detailed discussion of the derivation of the mixed Poisson distribution. In Chapter III, moment estimators for a mixture of two Poisson distributions are obtained and their asymptotic properties discussed. Chapter IV, contains a variety of results which are obtained as a by product of our approach.

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CHAPTER I

Introduction:

Mixtures of distributions have received an increasing amount of attention in the recent statistical literature, partly because of interest in the mathematical aspects of mixtures and partly because of the considerable number of applied problems in which mixtures are encountered.

The dissection of mixed frequency distribution is often very complicated [16]. This is certainly true for a mixture of two normal distributions, which was studied by Karl Pearson [22]; perhaps the earliest investigation of the dissection problem. Pearson was led to an equation of ninth degree the setting up and solution of which involved a tremendous amount of calculation. This calculation could doubtless be performed rather easily to-day by a high speed computer. (But when his paper was published in 1894, it was extremely laborious.)

Mixtures of distributions present two types of problems. The first is the problem of identifiability, that is, given that a distribution function F is a probability mixture of distribution functions belonging to some family $\mathcal{F}$, is the mixture unique?

This topic has been dealt with quite extensively in recent papers by Robbins [27], Teicher [30] and others. The second problem is

that of estimating the parameters of the individual distribution functions comprising the mixture (and mixing measure). This is clearly possible only if the given mixture is identifiable. (Pearson [22], C. R. Rao [23] and [24].)

Gumbel [15] solved the dissection problem for frequency distributions which are the sums of two asymmetrical exponential distributions, or of mixed Poisson distributions but in both cases, he restricted himself to the case in which the proportion contributed to the sum by each component is known.

Schilling [29] discussed extensively Chi-square test of the hypothesis that an observed distribution corresponds to

(1) a Poisson distribution,

(2) a weighted average of mixed Poisson distributions.

Applications are made to the examples given by F. Thorndike [33].

In 1961, Gumbel's theory was extended by Rider [25] to mixed exponential distributions and mixed-Poisson distributions.

CHAPTER II

2.1 Introduction:

In this Chapter, we will discuss the method of extimating the parameters of mixed distributions by using sample moments. This method has been outlined by Gumbel [15], who discussed the general dissection problem. He has shown how the method of moments can be used to estimate the parameters of a mixed distribution. He applied the method to mixed exponential and mixed Poisson distributions, but assumed that the proportion p is known. Gumbel's theory was extended by Rider [25] to a mixture of two exponential distributions. The same method is applied here to a mixture of two Poisson distributions.

Let us consider a population that is a mixture of two Poisson populations. Such an example would be a batch of manufactured articles mixed from two lots with different fractions defective. Another example would be the counts of bacterial colonies on the squares of a hemocytometer where different types of bacteria have been mixed.

To be definite, let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables, each having the distribution

$$P(X_i = x) = p \frac{\lambda^x e^{-\lambda}}{x!} + (1-p) \frac{\mu^x e^{-\mu}}{x!}$$

$$\text{if } x = 0, 1, 2, \ldots$$

$$= 0 \quad \text{otherwise} \tag{2.1.1}$$

where

$$0 < p < 1 \; ; \qquad \lambda,\mu > 0 \quad .$$

We shall refer to the distribution (2.1.1) as the mixed-Poisson distribution. λ and μ are the parameters of two Poisson distributions, which have been mixed in the unknown proportions of p and $(1-p)$.

2.2 Mathematical Properties:

When investigating random variables which take on only the integers $x = 0, 1, 2, \ldots$ it is simpler to deal with probability generating functions than with characteristic functions.

Let X be a random variable and let

$$p_x = P(X = x) \qquad (x = 0,1,2,\ldots) \quad ,$$

then

$$\sum_x p_x = 1 \quad .$$

The probability generating function (p.g.f.) of X is defined by

$$\psi(s) = \sum_x p_x s^x \qquad (-1 \leq s \leq 1)$$

we notice that

$$\psi(1) = \sum_x p_x = 1 \tag{2.2.1}$$

Hence the series on the right-hand side is absolutely and uniformly convergent in the interval $|s| \leq 1$. Thus the generating function is continuous. It determines the probability function uniquely, since $\psi(s)$ can be represented in a unique way as a power series of the form

$$\psi(s) = \sum_{x=0}^{\infty} p_x s^x \qquad (-1 \leq s \leq 1)$$

$$(\sum_x p_x = 1)$$

$$P(X = x) = p \frac{\lambda^x e^{-\lambda}}{x!} + (1-p) \frac{\mu^x e^{-\mu}}{x!} \quad (x = 0,1,2,\ldots) \; .$$

The probability generating function (p.g.f.) of the mixed-Poisson distribution is given by

$$\begin{aligned} \psi(s) &= \sum_x p_x s^x \\ &= \sum_x s^x \left[p \frac{\lambda^x e^{-\lambda}}{x!} + (1-p) \frac{\mu^x e^{-\mu}}{x!} \right] \\ &= p\, e^{-\lambda(1-s)} + (1-p)\, e^{-\mu(1-s)} \; . \end{aligned}$$

The moments of the random variable X can be determined by the derivatives at the point 1 of the genreating function. Let us, for example, determine the moments of the first and second order. We have

$$\psi'(s) = \sum_x x p_x s^{x-1}$$

$$\psi''(s) = \sum_x x(x-1)\, p_x s^{x-2} \; .$$

Hence

$$\psi'(1) = \sum_x x p_x = E(X)$$

$$\psi''(1) = \sum_x x(x-1)\, p_x = E(X^2) - E(X)$$

we then obtain

$$E(X^2) = \psi''(1) + \psi'(1)$$

2.3 <u>The Moment Generating Function of The Mixed-Poisson Distribution</u>

Since

$$e^{tx} = 1 + \frac{(tx)}{1!} + \frac{(tx)^2}{2!} + \ldots = \sum_{i=o}^{\infty} \frac{t^i x^i}{i!} ,$$

$$E\{e^{tx}\} = \sum_{i=o}^{\infty} \frac{\mu_i^i t^i}{i!}$$

where

$$\mu_o' = 1 .$$

The function

$$m(t) = E(e^{tx})$$

is called the moment generating function (m.g.f.) of x. If $m(t)$ be differentiated k-times with respect to t and then evaluated at $t = o$,

we note that

$$\left.\frac{\partial^k m(t)}{\partial t^k}\right|_{t=o} = \mu_k'$$

to obtain the moment genrating function of $x-\mu$, we consider

$$e^{t(x-\mu)} = 1 + \frac{t(x-\mu)}{1!} + \frac{t^2(x-\mu)^2}{2!} + \ldots$$

then

$$E\left\{e^{t(x-\mu)}\right\} = 1 + \sum_{i=1}^{\infty} \mu_i \frac{t^i}{i!}$$

if we set,

$$M(t) = E\left\{e^{t(x-\mu)}\right\} ,$$

then

$$\mu_k = \left.\frac{\partial^k M(t)}{\partial t^k}\right|_{t=o} .$$

The moment generating function of any function $\theta(x)$ may be defined as

$$M(t) = E\left\{e^{t\theta(x)}\right\} .$$

Since it may easily be seen that

$$\left.\frac{\partial^k M(t)}{\partial t^k}\right|_{t=o} = \mu_k'$$

where μ' is the k-th moment of $\theta(x)$ about the origin.

In the case of the mixed-Poisson distribution, we have

$$m(t) = E\{e^{tx}\}$$

$$= \sum_{x=0}^{\infty} e^{tx} p_x$$

$$= \sum_{x=0}^{\infty} e^{tx} \left[p \frac{\lambda^x e^{-\lambda}}{x!} + (1-p) \frac{\mu^x e^{-\mu}}{x!} \right]$$

$$= p e^{-\lambda(1-e^t)} + (1-p) e^{-\mu(1-e^t)} \quad .$$

Thus

$$m(t) = p e^{-\lambda(1-e^t)} + (1-p) e^{-\mu(1-e^t)} \tag{2.3.1}$$

and putting $1-e^t = \delta$ in (2.3.1), we get

$$m(t) = p e^{-\lambda\delta} + (1-p) e^{-\mu\delta}$$

we may obtain the following relation between $M(t)$ and $m(t)$:

$$M(t) = E[e^{t(x-\mu)}]$$

$$= e^{-t\mu} E[e^{tx}]$$

$$= e^{-t\mu} m(t)$$

Hence

$$M(t) = e^{-t\mu} m(t) \quad .$$

2.4 Mean and Variance of The Mixed-Poisson Distribution

$$E(X) = \sum_{x=0}^{\infty} x\, P_x$$

$$= \sum_{x=0}^{\infty} x \left[p \frac{\lambda^x e^{-\lambda}}{x!} + (1-p) \frac{\mu^x e^{-\mu}}{x!} \right]$$

$$= \sum_{x=0}^{\infty} x\, p \frac{\lambda^x e^{-\lambda}}{x!} + (1-p) \sum_{x=0}^{\infty} x \frac{\mu^x e^{-\mu}}{x!}$$

$$= p\,\lambda + (1-p)\,\mu \quad .$$

Hence

$$\text{mean} = p\,\lambda + (1-p)\,\mu \quad .$$

$$m_2 = E(X^2)$$

$$= \sum_{x=0}^{\infty} x^2\, p_x$$

$$= \sum_{x=0}^{\infty} [x(x-1)+x] \left[p \frac{\lambda^x e^{-\lambda}}{x!} + (1-p) \frac{\mu^x e^{-\mu}}{x!} \right]$$

$$= \sum_{x=0}^{\infty} x(x-1) \left[p \frac{\lambda^x e^{-\lambda}}{x!} + (1-p) \frac{\mu^x e^{-\mu}}{x!} \right] +$$

$$+ \sum_{x=0}^{\infty} x \left[p \frac{\lambda^x e^{-\lambda}}{x!} + (1-p) \frac{\mu^x e^{-\mu}}{x!} \right]$$

$$= p\,\lambda^2 + (1-p)\,\mu^2 + p\,\lambda + (1-p)\,\mu$$

Hence,

$$\mu_2 = p\,\lambda^2 + (1-p)\,\mu^2 + p\,\lambda + (1-p)\,\mu -$$

$$- [p\,\lambda + (1-p)\,\mu]^2$$

$$= p(1-p)\,[(\lambda-\mu)^2 + \frac{\lambda}{(1-p)} + \frac{\mu}{p}\,] \quad ,$$

similarly

$$m_3 = E(X^3)$$

$$= \sum_{x=0}^{\infty} x^3 p_x$$

$$= \sum_{x=0}^{\infty} [x(x-1)(x-2)+3x(x-1)+x]\; p_x$$

$$= p\,\lambda^3 + (1-p)\,\mu^3 + 3p\,\lambda^2 + 3(1-p)\,\mu^2 +$$

$$+ p\,\lambda + (1-p)\,\mu \quad .$$

Hence,

$$m_3 = p(\lambda+3\lambda^2+\lambda^3) + (1-p)(\mu+3\mu^2+\mu^3) \quad .$$

In the mixed-Poisson distribution, we can easily see that the two requirements for a function of a random variable to be a probability density function are satisfied by noting that

$$(1) \qquad p\,\frac{\lambda^x e^{-\lambda}}{x!} + (1-p)\,\frac{\mu^x e^{-\mu}}{x!} > 0 \qquad \text{if} \quad x = 0,1,2,\ldots \quad .$$

$$(2) \quad \sum_{x=0}^{\infty} p(x;\ \lambda,\mu,p) = 1$$

where $p(x;\ \lambda,\mu,p)$ is the probability density function of the mixed-Poisson distribution. The comulative distribution function of X is given by

$$F(x;\ \lambda,\mu) = \sum_{\gamma=0}^{x} p(\gamma;\ \lambda,\mu,p) \qquad \text{if} \quad x = 0,1,2,$$

2.5 Method of Moments:

A method of estimating the parameters of mixed disbributions by using sample moments has been outlined by Gumbel [15], who considered mixed exponential and mixed-Poisson distributions. The method is applied here for estimating the parameters of the mixed-Poisson distribution.

Consider the frequency distribution of the mixed-Poisson distribution

$$f(x) = p\,\frac{\lambda^x e^{-\lambda}}{x!} + (1-p)\,\frac{\mu^x e^{-\mu}}{x!} \tag{2.5.1}$$

Let m_1', m_2' and m_3' be the first three moments about the origin, of a sample from mixed-Poisson distribution, and let p^*, λ^* and μ^* denote the estimators of p, λ, and μ respectively, obtained by the method of moments.

Then

$$p^* \lambda^* + (1-p^*)\,\mu^* = m_1' \tag{2.5.2}$$

$$p^*(\lambda^* + \lambda^{*2}) + (1-p^*)(\mu^* + \mu^{*2}) = m_2' \qquad (2.5.3)$$

$$p^*(\lambda^* + 3\lambda^{*2} + \lambda^{*3}) + (1-p^*)(\mu^* + 3\mu^{*2} + \mu^{*3}) = m_3' \qquad (2.5.4)$$

or

$$p^* (\lambda^* - \mu^*) = m_1' - \mu^* \quad . \qquad (2.5.5)$$

From (2.5.3)

$$p^* (\lambda^* - \mu^*)(1 + \lambda^* + \mu^*) = m_2' - \mu^* - \mu^{*2} \quad . \qquad (2.5.6)$$

From (2.5.4)

$$p^* (\lambda^* - \mu^*)[1 + 3(\lambda^* + \mu^*) + \lambda^{*2} + \lambda^* \mu^* + \mu^{*2}] =$$

$$= m_3' - \mu^* - 3\mu^{*2} - \mu^{*3} \quad . \qquad (2.5.7)$$

Substituting the expression for $p^* (\lambda^* - \mu^*)$ from (2.5.5) into (2.5.6) and (2.5.7) leads to the following equations.

$$\frac{m_1' - \mu^*}{\lambda^* - \mu^*} [\lambda^* + \lambda^{*2}] + \left[1 - \frac{m_1' - \mu^*}{\lambda^* - \mu^*}\right] (\mu^* + \mu^{*2}) = m_2'$$

$$(m_1' - \mu^*)(\lambda^* + \lambda^{*2}) + (\lambda^* - m_1')(\mu^* + \mu^{*2}) = m_2' (\lambda^* - \mu^*)$$

$$m_1' (\lambda^* - \mu^*) + m_1' (\lambda^{*2} - \mu^{*2}) - \lambda^* \mu^* (\lambda^* - \mu^*) = m_2' (\lambda^* - \mu^*)$$

$$m_1' (\lambda^* - \mu^*) + m_1' (\lambda^* - \mu^*)(\lambda^* + \mu^*) - \lambda^* \mu^* (\lambda^* - \mu^*) = m_2' (\lambda^* - \mu^*)$$

$$m_1' + m_1' (\lambda^* + \mu^*) - \lambda^* \mu^* = m_2'$$

$$m_1' [1 + (\lambda^* + \mu^*)] - \lambda^* \mu^* = m_2'$$

$$m_1' [1 + S^*] - P^* = m_2' \qquad (2.5.8)$$

where

$$S^* = \lambda^* + \mu^*$$

and

$$P^* = \lambda^* \mu^* \quad .$$

From equation (2.5.7)

$$p^* (\lambda^* + 3\lambda^{*2} + \lambda^{*3}) + (1-p^*)(\mu^* + 3\mu^{*2} + \mu^{*3}) = m_3'$$

$$(m_1' - \mu^*)(\lambda^* + 3\lambda^{*2} + \lambda^{*3}) + (\lambda^* - m_1')(\mu^* + 3\mu^{*2} + \mu^{*3}) = m_3' (\lambda^* - \mu^*)$$

$$m_1'(\lambda^* + 3\lambda^{*2} + \lambda^{*3}) - \mu^* (\lambda^* + 3\lambda^{*2} + \lambda^{*3}) + \lambda^* (\mu^* + 3\mu^{*2} + \mu^{*3}) -$$
$$- m_1' (\mu^* + 3\mu^{*2} + \mu^{*3}) = m_3' (\lambda^* - \mu^*)$$

$$m_1' + 3m_1' (\lambda^* + \mu^*) + m_1' (\lambda^{*2} + \lambda^* \mu^* + \mu^{*2}) - 3\lambda^* \mu^* -$$
$$- \lambda^* \mu^* (\lambda^* + \mu^*) = m_3'$$

$$m_1' [1 + 3(\lambda^* + \mu^*) + (\lambda^{*2} + \lambda^* \mu^* + \mu^{*2})] - 3\lambda^* \mu^* -$$
$$- \lambda^* \mu^* (\lambda^* + \mu^*) = m_3'$$

$$m_1' [1 + 3S^* + S^{*2} - P^*] - 3P^* - S^* P^* = m_3' \tag{2.5.9}$$

where

$$S^* = \lambda^* + \mu^*$$

$$P^* = \lambda^* \mu^* \quad .$$

The solutions of (2.5.8) and (2.5.9) are readily found to be

$$S^* = \frac{m_1'^2 + 2m_1' - m_1' m_2' - 3m_2' + m_3'}{m_2' - m_1'^2 - m_1'} \tag{2.5.10}$$

and

$$P^* = \frac{m_1'^2 - m_1' m_2' + m_1' m_3' - m_2'^2}{m_2' - m_1'^2 - m_1'} \tag{2.5.11}$$

The estimates λ^* and μ^* are

$$\frac{1}{2} S^* \pm \frac{1}{2} [S^{*2} - 4P^*]^{1/2} \quad . \tag{2.5.12}$$

Finally, the estimator p^* can be found from (2.5.5)

Remarks: (see Rider [25])

(1) The roots of the quadratic are λ^* and μ^*, it being immaterial which root is designated λ^* and which μ^*. That is, the estimate p^* of the proportion p, obtained by substituting λ^* and μ^* respectively in

(2.5.5), will refer to the component having λ as a parameter and $(1-p^*)$ will refer to the other component.

(2) From continuity considerations, it is seen that imaginary roots may occur with positive probability. However, if $\lambda \neq \mu$, the proposed estimators are consistent, and the probability that $\lambda^* > 0$, $\mu^* > 0$, $0 \leq p^* < 1$ approaches 1 as $n \to \infty$. This follows from the fact that, in this case, the estimators, regarded as functions of (m_1', m_2', m_3'), are continuous at the point (μ_1', μ_2', μ_3'), where the μ_i''s are the population moments, and that $\lambda^* > 0$, $0 \leq p^* < 1$ if (m_1', m_2', m_3') is sufficiently close to (μ_1', μ_2', μ_3').

(3) If $\lambda = \mu = \nu$, the behavior of the estimators changes radically. Some discussion of the variances of the three estiamtors is of course in order. It seems that the calculation of these variances, even in asymptotic form, would not only be a difficult task but would lead to somewhat complicated expressions. However, to simplify matters and to give some idea of the reliability of the estimators λ^* and μ^*, it will be temporarily assumed that p is known. With this assumption, only two sample moments are needed to estimate λ and μ.

(4) From equations (2.5.2) and (2.5.3) it is found that

$$\lambda^* = m_1' + \left(\frac{q}{p}\right)^{1/2} [m_2' - m_1'^2 - m_1']^{1/2}$$

where, as usual, $q = 1-p$ and μ^* is equal to the same expression with a minus sign between the two terms on the right-hand side.

(5) The plus sign is used if $\lambda \geq \mu$, the negative sign is used if $\lambda \leq \mu$. The first pair of estimators are consistent when $\lambda \geq \mu$, the second pairs when $\lambda \leq \mu$. Thus, here the estimators are consistent if $\lambda = \mu$, but in this case the rate of approach to the limit is $n^{-1/4}$ as compared with $n^{-1/2}$ for $\lambda \neq \mu$. Also if $\lambda = \mu$ the probability that the estimators are real does not approach 1 as n approaches infinity, although the imaginary parts converge to zero in probability. It is assumed that $\lambda \neq \mu$. If it is not known whether $\lambda > \mu$ or $\lambda < \mu$, then it is also not known which pair of estimators is consistent, that is, which pair may be expected to be close to the true value when the sample is large. (Rider [25].)

2.6 Illustrative Example

E. L. Grant [11], has given the example, shown in Table I below, of the distribution of numbers of surface defects in 2,000 pieces of enameled wave. The distribution is composed of 1,000 pieces from a Poisson distribution having $\lambda = 0.5$ and 1,000 pieces from a Poisson distribution having $\mu = 1.05$

TABLE I

Number of defects	0	1	2	3	4	5	6	7	TOTAL
Frequency	830	638	327	137	49	15	3	1	2,000

Simple claculations yield:

$$m_1' = 1 \ , \quad m_2' = 2.2475$$

$$m_3' = 6.4775 \quad .$$

From (2.5.1) the following are obtained:

$$S^* = 1.9697$$

and from (2.5.11) or (2.5.8),

$$P^* = 0.7222 \quad .$$

It follows that

$$\lambda^* = 0.4948 \ ; \quad \mu^* = 1.4713$$

$$p^* = 0.4844 \ ; \quad (1-p^*) = 0.5156 \quad .$$

2.7 Asymptotic Variances of the Poisson Estimators

Rider [26] discusses the asymptotic variances of the Poisson estimators, which are given below:

For purposes of simplification, it will be assumed that the proportion p in the mixed-Poisson distribution is known. Then λ^* and μ^* can be obtained by using only the first two sample moments. It is readily found that

$$\lambda^* = m_1' + \left(\frac{q}{p}\right)^{1/2} [m_2' - m_1'^2 - m_1']^{1/2}$$

and (2.7.1)

$$\mu^* = m_1' - \left(\frac{q}{p}\right)^{1/2} [m_2' - m_1'^2 - m_1']^{1/2}$$

The asymptotic variances of λ^* and μ^* may be found by the use of a formula given by Cramer [10].

In the notation of this thesis, this formula is

$$\text{var }(\lambda^*) = \mu_2(m_1')\left(\frac{\partial\lambda^*}{\partial m_1'}\right)^2 + 2\mu_{11}(m_1', m_2')\left(\frac{\partial\lambda^*}{\partial m_1'}\right)\left(\frac{\partial\lambda^*}{\partial m_2'}\right) + $$
$$+ \mu_2(m_2')\left(\frac{\partial\lambda^*}{\partial m_2'}\right)^2$$

and

$$\text{var }(\mu^*) = \mu_2(m_1')\left(\frac{\partial\mu^*}{\partial m_1'}\right)^2 + 2\mu_{11}(m_1', m_2')\left(\frac{\partial\mu^*}{\partial m_1'}\right)\left(\frac{\partial\mu^*}{\partial m_2'}\right) + $$
$$+ \mu_2(m_2')\left(\frac{\partial\mu^*}{\partial m_2'}\right)^2 \qquad (2.7.2)$$

Here $\mu_2(m_1')$ and $\mu_2(m_2')$ are the variances of m_1' and m_2' respectively, $\mu_{11}(m_1', m_2')$ is the covariance of these two moments, and the partial derivatives are to be evaluated at the point

$$\begin{aligned} m_1' &= p\lambda + q\mu \\ m_2' &= p(\lambda + \lambda^2) + q(\mu + \mu^2) \\ m_3' &= p(\lambda + 3\lambda^2 + \lambda^3) + q(\mu + 3\mu^2 + \mu^3) \quad . \end{aligned} \tag{2.7.3}$$

The values of the coefficients of the partial derivatives in (2.7.2) can be obtained by using a formula given by Kendall [18]. It is found that:

$$\begin{aligned} \mu_2(m_1') &= n^{-1}(\mu_2' - \mu_1'^2) \\ &= n^{-1}[p\lambda + q\mu + pq(\lambda-\mu)^2] \end{aligned} \tag{2.7.4}$$

where $p + q = 1$

$$\begin{aligned} \mu_{11}(m_1', m_2') &= n^{-1}(\mu_3' - \mu_1' \mu_2') \\ &= n^{-1}[p\lambda + q\mu + p(3-p)\lambda^2 - 2pq\lambda\mu + \\ &\quad + q(3-q)\mu^2] \end{aligned} \tag{2.7.5}$$

$$\begin{aligned} \mu_2(m_2') &= n^{-1}(\mu_4' - \mu_2'^2) \\ &= n^{-1}[p\lambda + q\mu + p(7-p)\lambda^2 - 2pq\lambda\mu + \\ &\quad + q(7-q)\mu^2 + 2p(3-p)\lambda^3 - 2pq\lambda^2\mu - \\ &\quad - 2pq\lambda\mu^2 + 2q(3-q)\mu^3 + pq(\lambda^2 - \mu^2)^2] \; . \end{aligned} \tag{2.7.6}$$

The partial derivatives needed are:

$$\frac{\partial\lambda^*}{\partial m_1'} = 1 - \frac{q^{1/2}(1+2m_1')}{2p^{1/2}(m_2'-m_1'^2-m_1')^{1/2}} \tag{2.7.7}$$

$$\frac{\partial\lambda^*}{\partial m_2'} = \frac{q^{1/2}}{2p^{1/2}(m_2'-m_1'^2-m_1')^{1/2}} \quad . \tag{2.7.8}$$

At the point (2.7.3) these derivatives have the values

$$\frac{\partial\lambda^*}{\partial m_1'} = -\frac{1+2\mu}{2p(\lambda-\mu)} \; ; \quad \frac{\partial\lambda^*}{\partial m_2'} = \frac{1}{2p(\lambda-\mu)} \quad . \tag{2.7.9}$$

Substituting from (2.7.4), (2.7.5), (2.7.6) and (2.7.9) into (2.7.2) and simplifying give, for the asymptotic variance of λ^* ,

$$[4np^2(\lambda-\mu)^2]^{-1}\,[2(p\lambda^2+q\mu^2) + 4p\lambda(\lambda-\mu)^2 - pq(\lambda-\mu)^4] \tag{2.7.10}$$

The asymptotic variance of μ^* can be obtained by interchanging p and q , λ and μ in (2.7.10) .

It is the persomal opinion of Rider [25] that data should not be assumed to have come from mixed-Poisson distribution until it has been determined that they have not come from a simple Poisson distribution. That is, the parameter λ of the distribution, should be estimated, following which, a Chi-square test should be made to see whether the data conform to this distribution. If the hypothesis that they came from a simple Poisson is rejected, a mixed-Poisson population may be assumed.

Of course, the Chi-square test may give the wrong conclusion, in which case it should be impossible to find, by the method under discussion, an estimate of λ . Even if the population is mixed and λ and μ are nearly equal, it might be difficult to obtain valid estimates of them.

CHAPTER III

ASYMPTOTIC BEHAVIOR OF THE MOMENT ESTIMATORS

3.0 Introduction

Blischke [3] has discussed the construction of moment estimators for the parameters of a mixture of two binomial distributions. He also computed the limiting distributions of the estimators and their asymptotic efficiency. In this chapter, construction of moment estimators for a mixed-Poisson distribution is given and their asymptotic properties discussed. The construction presented here parallels that of Blischke.

3.1 Construction of Moment Estimators for a Mixture of Two-Poisson Distributions

Let $X_1, X_2, \ldots, X_n$ be independent and identically distrubuted random variables each having the distribution

$$P(X_i = x) = p\,\frac{\lambda^x e^{-\lambda}}{x!} + (1-p)\,\frac{\mu^x e^{-\mu}}{x!} \qquad (3.1.1)$$

$$\text{if } x = 0,1,2,\ldots$$

$$= 0 \quad \text{otherwise.} \quad (0 < p < 1\ ;\ \lambda,\mu > 0)$$

where without any loss of generality, we let $\lambda > \mu$. Equation (3.1.1) is a mixture of two Poisson distributions. The estimators for the parameters λ, μ, and p may be constructed as functions of sample

factorial moments. It should be noted that in the following construction, as in the example considered by Rider [25], $\lambda < \mu$ is essential. The estimators do not have the stated properties if $\lambda = \mu$. (Note that it is actually required only that $\lambda \neq \mu$; the particular assumption that $\lambda < \mu$ is for notational convenience.)

The k^{th} factorial mement of x can be written as

$$m_{[k]} = p\lambda^k + (1-p)\mu^k \tag{3.1.2}$$

$$= p(\lambda^k - \mu^k) + \mu^k .$$

To obtain estimators for the three parameters λ, μ, and p we equate esimates f_1, f_2, and f_3 of the first three factorial moments $m_{[1]}$, $m_{[2]}$, and $m_{[3]}$ respectively of a random sample from a distribution with probability function [3.1.1) to the corresponding moments (3.1.2). Let $\hat{p}$, $\hat{\lambda}$, $\hat{\mu}$ are the estimators of p, λ, μ respectively Thus we have

$$f_1 = \hat{p}\,(\hat{\lambda} - \hat{\mu}) + \hat{\mu}$$

$$f_2 = \hat{p}\,(\hat{\lambda}^2 - \hat{\mu}^2) + \hat{\mu}^2$$

$$f_3 = \hat{p}\,(\hat{\lambda}^3 - \hat{\mu}^3) + \hat{\mu}^3$$

where

$$f_k = \sum_{x=o}^{R} x(x-1) \ldots (x-k+1) \frac{n_x}{n} \quad .$$

Where R is the largest observed value of x, n_x is the sample frequency of x, and n is the total sample size;

$$\left(n = \sum_{x=o}^{R} n_x \right) .$$

Now

$$f_k = \frac{1}{n} \sum_{i=1}^{n} x_i(x_i-1) \ldots (x_i-k+1) \quad .$$

It is easily seen in fact that

$$f_k = \hat{p}\,\hat{\lambda}^k + (1-\hat{p})\,\hat{\mu}^k \quad . \tag{3.1.3}$$

In constructing estimators, the moments given in (3.1.3) are considered as equations in three unknowns p, λ, μ . Any three such equations may be solved for the three parameters. We shall consider f_1, f_2, and f_3 . Note that by equation (3.1.3),

$$f_2 - f_1^2 = \hat{p}(1-\hat{p})(\hat{\lambda}-\hat{\mu})^2$$

and

$$f_3 - f_1 f_2 = \hat{p}(1-\hat{p})(\hat{\lambda}+\hat{\mu})(\hat{\lambda}-\hat{\mu})^2 \quad ,$$

so that

$$\hat{\lambda}+\hat{\mu} = \frac{f_3 - f_1 f_2}{f_2 - f_1^2}$$

$$= S^* \text{, say .}$$

Now from f_1 ,

$$\hat{p} = \frac{f_1 - \hat{\mu}}{\hat{\lambda} - \hat{\mu}} \tag{3.1.4}$$

which, when substituted into f_2 , yields

$$f_2 = (f_1 - \hat{\mu}) S^* + \hat{\mu}^2 \quad .$$

Thus

$$\hat{\mu}^2 - S^* \hat{\mu} + S^* f_1 - f_2 = 0 \tag{3.1.5}$$

Solving for $\hat{\lambda}$ instead of $\hat{\mu}$ yields

$$\hat{\lambda}^2 - S^* \hat{\lambda} + S^* f_1 - f_2 = 0 \quad .$$

Thus the restriction that $\lambda < \mu$ results in the unique solution

$$\hat{\mu}, \hat{\lambda} = \frac{1}{2} S^* \pm \frac{1}{2} [S^{*^2} - 4S^* f_1 + 4f_2]^{1/2} \tag{3.1.6}$$

which together with equation (3.1.4) expresses $\hat{\lambda}$, $\hat{\mu}$, and $\hat{p}$ as functions of f_1 f_2, and f_3 respectively.

3.2 Asymptotic Behavior of the Moment Estimators

The asymptotic joint distribution of λ^*, μ^*, p^* is obtained by an extension of a result given by Cramér [10]. Cramér proves a theorem concerning the limiting distribution of a function of two central moments. Extended to several functions and to functions of factorial moments, the result is the following:

THEOREM: Let $H_i(F_{j_1}, \ldots, F_{j_k})$, $i = 1,\ldots,r$, be r functions of K-sample factorial moments, continuous in some neighbourhood of $(F_{j_1},\ldots,F_{j_k}) = (f_{j_1},\ldots,f_{j_k}) = f$, say, and having continuous first and second order partial derivatives with respect to the arguments $F_{j_1},\ldots,F_{j_k}$. Then the random variables $H_1,\ldots,H_r$ are asymptotically jointly normally distributed with respective means $H_1(f_{j_1},\ldots,f_{j_k}),\ldots,$ $H_r(f_{j_1},\ldots,f_{j_k})$ and covariance matrix $\Sigma = (\sigma_{ii'})$ having entries

$$\sigma_{ii'} = \sum_{t=1}^{k} \mu_2(F_{j_t}) \left(\frac{\partial H_i}{\partial F_{j_t}} \Big|_f \right) \left(\frac{\partial H_{i'}}{\partial F_{j_t}} \right) \Big|_f$$

$$+ \sum_{\substack{s,t=1 \\ s \neq t}}^{k} \mu_{11}(F_{j_s}, F_{j_t}) \left(\frac{\partial H_i}{\partial F_{j_s}} \Big|_f \right) \left(\frac{\partial H_{i'}}{\partial F_{j_t}} \Big|_f \right)$$

Cramér proves the theorem for central moments and for $r = 1$. For $r = 1$, the argument given by Cramér applies directly in the case of factorial moments since the convergence properties of central moments

required in the proof hold also for factorial moments. The extension to $r > 1$ is obvious in either case.

The conditions of the theorem are easily seen to hold for $H_1(F_1, F_2, F_3) = \lambda^*$, $H_2(F_1, F_2, F_3) = \mu^*$ and $H_3(F_1, F_2, F_3) = p^*$. Thus λ^*, μ^*, p^* are asymptotically jointly normally distributed with means λ, μ, and p respectively. To compute the covariance matrix of the asymptotic normal distribution, the covariance matrix of (F_1, F_2, F_3) and the first order partial derivative $\partial H_i/\partial F_j$ evaluated at $f = (f_1, f_2, f_3)$ are required for $i,j = 1,2,3$.

The moment generating function of the distribution [Equation (3.1.1)] is

$$m(t) = p\, e^{-\lambda(1-e^t)} + (1-p)\, e^{-\mu(1-e^t)} \tag{3.2.1}$$

From equation (3.2.1) the necessary moments of the F_i are obtained. By equation (3.1.3) these can be written as

$$\mu_2(F_1) = \frac{1}{mn}\,[f_1+(n-1)f_2] - \frac{1}{m}\,(f_1)^2$$

$$\mu_2(F_2) = \frac{1}{mn(n-1)}\,[2f_2+4(n-2)f_3+(n-2)(n-3)f_4] - \frac{1}{m}\,(f_2)^2$$

$$\mu_2(F_3) = \frac{1}{mn(n-1)(n-2)}\,[6f_3+18(n-3)f_4+9(n-3)(n-4)f_5$$

$$+ (n-3)(n-4)(n-5)f_6] - \frac{1}{m}\,(f_3)^2$$

$$\mu_{11}(F_1,F_2) = \frac{1}{mn}\,[2f_2+(n-2)f_3] - \frac{1}{m}\,f_1f_2$$

$$\mu_{11}(F_1,F_3) = \frac{1}{mn}\,[3f_3+(n-3)f_4] - \frac{1}{m}\,f_1f_3$$

$$\mu_{11}(F_2,F_3) = \frac{1}{mn(n-1)}\,[6f_3+6(n-3)f_4+(n-3)(n-4)f_5] - \frac{1}{m}\,f_2f_3$$

The required partial derivatives are not difficult to compute:

$$\left.\frac{\partial H_1}{\partial F_1}\right|_f = \left.\left[\frac{\partial H_1}{\partial A}\cdot\frac{\partial A}{\partial F_1} + \frac{\partial H_1}{\partial F_1}\right]\right|_f$$

$$= \frac{\mu(2\lambda + \mu)}{p(\lambda - \mu)^2} \quad .$$

By symmetry,

$$\left.\frac{\partial H_2}{\partial F_1}\right|_f = \frac{\lambda(2\mu+\lambda)}{(1-p)(\lambda-\mu)^2} \quad .$$

Also,

$$\frac{\partial H_3}{\partial F_1}\bigg|_f = \left[\frac{\partial H_3}{\partial H_1}\frac{\partial H_1}{\partial F_1} + \frac{\partial H_3}{\partial H_2}\frac{\partial H_2}{\partial F_1} + \frac{\partial H_3}{\partial F_1}\right]\bigg|_f$$

$$= -\frac{f_1-\mu}{(\lambda-\mu)^2}\cdot\frac{\mu(\mu+2\lambda)}{p(\lambda-\mu)^2} + \frac{(f_1-\mu)-(\lambda-\mu)}{(\lambda-\mu)^2}\cdot\frac{\lambda(\lambda+2\mu)}{(1-p)(\lambda-\mu)^2} + \frac{1}{\lambda-\mu}$$

$$= -\frac{6\lambda\mu}{(\lambda-\mu)^3}, \qquad \text{etc.}\ .$$

In matrix form the derivatives are

$$\frac{\partial H_i}{\partial F_j}\bigg|_{f_1,f_2,f_3} = \frac{1}{(\lambda-\mu)^2}\begin{bmatrix} \frac{\mu(\mu+2\lambda)}{p} & -\frac{\lambda+2\mu}{p} & \frac{1}{p} \\ \frac{\lambda(\lambda+2\mu)}{1-p} & -\frac{2\lambda+\mu}{1-p} & \frac{1}{1-p} \\ \frac{6\lambda\mu}{\mu-\lambda} & -\frac{3(\lambda+\mu)}{\mu-\lambda} & \frac{2}{\mu-\lambda} \end{bmatrix}$$

Thus by Cramér's theorem the variance in the limiting distribution of λ^* is

$$\sigma^2_{\lambda^*} = \mu_2(F_1)\left[\frac{\partial H_1}{\partial F_1}\bigg|_f\right]^2 + \mu_2(F_2)\left[\frac{\partial H_1}{\partial F_2}\bigg|_f\right]^2 +$$

$$+ \mu_2(F_3)\left[\frac{\partial H_1}{\partial F_3}\bigg|_f\right]^2 + 2\mu_{11}(F_1,F_2)\left[\frac{\partial H_1}{\partial F_1}\bigg|_f\frac{\partial H_1}{\partial F_2}\bigg|_f\right]$$

$$+ 2\mu_{11}(F_1,F_3)\left[\frac{\partial H_1}{\partial F_1}\bigg|_f\frac{\partial H_1}{\partial F_3}\bigg|_f\right] + 2\mu_{11}(F_2,F_3)\left[\frac{\partial H_1}{\partial F_2}\bigg|_f\frac{\partial H_1}{\partial F_3}\bigg|_f\right]$$

$$= \left[\frac{\mu^2+2\lambda\mu}{p(\lambda-\mu)}\right]^2\left\{\frac{1}{nm}[p\lambda+(1-p)\mu] + \right.$$

$$+ \frac{n-1}{nm} [p\lambda^2+(1-p)\mu^2] - \frac{1}{m} [p\lambda+(1-p)\mu^2] \Big\}$$

$$+ \left[\frac{\lambda+2\mu}{p(\lambda-\mu)^2} \right]^2 \Big\{ \frac{(n-2)(n-3)}{mn(n-1)} [p\lambda^4+(1-p)\mu^4]$$

$$+ \frac{4(n-2)}{mn(n-1)} [p\lambda^3+(1-p)\mu^3] + \frac{2}{mn(n-1)} [p\lambda^2+(1-p)\mu^2]$$

$$- \frac{1}{m} [p\lambda^2+(1-p)\mu^2]^2 \Big\}$$

$$+ \left[\frac{1}{p(\lambda-\mu)^2} \right]^2 \Big\{ \frac{(n-3)(n-4)(n-5)}{mn(n-1)(n-2)} [p\lambda^6+(1-p)\mu^6]$$

$$+ \frac{9(n-3)(n-4)}{mn(n-1)(n-2)} [p\lambda^5+(1-p)\mu^5]$$

$$+ \frac{18(n-3)}{mn(n-1)(n-2)} [p\lambda^4+(1-p)\mu^4]$$

$$+ \frac{6}{mn(n-1)(n-2)} [p\lambda^3+(1-p)\mu^3] - \frac{1}{m} [p\lambda^3+(1-p)\mu^3]^2 \Big\}$$

$$- \frac{2\mu(\mu+2\lambda)(\lambda+2\mu)}{p^2(\lambda-\mu)^4} \Big\{ \frac{2}{mn} [p\lambda^2+(1-p)\mu^2]$$

$$+ \frac{n-2}{mn} [p\lambda^3 + (1-p)\mu^3] - \frac{1}{m} [p\lambda+(1-p)\mu][p\lambda^2+(1-p)\mu^2] \Big\}$$

$$+ \frac{2\mu(\mu+2\lambda)}{p^2(\lambda-\mu)^4} \Big\{ \frac{3}{mn} [p\lambda^3+(1-p)\mu^3] + \frac{n-3}{mn} [p\lambda^4+(1-p)\mu^4]$$

$$- \frac{1}{m} [p\lambda+(1-p)\mu][p\lambda^3+(1-p)\mu^3] \Big\}$$

$$- \frac{2(\lambda+2\mu)}{p^2(\lambda-\mu)^4} \Big\{ \frac{6}{mn(n-1)} [p\ \lambda^3+(1-p)\mu^3]$$

$$+ \frac{6(n-3)}{mn(n-1)} [p\lambda^4+(1-p)\mu^4] + \frac{(n-3)(n-4)}{mn(n-1)} [p\lambda^5+(1-p)\mu^5]$$

$$- \frac{1}{m} [p\lambda^2+(1-p)\mu^2][p\lambda^3+(1-p)\mu^3] \Big\} .$$

To simplify this expression, write

$$(n-2)(n-3) = n(n-1) - 4(n-1)+2$$

$$(n-3)(n-4) = (n-1)(n-2) - 4(n-2)+2$$

$$= n(n-1) - 6(n-1)+6$$

$$(n-3)(n-4)(n-5) = n(n-1)(n-2) - 9(n-1)(n-2) + 18)n-2)-6 ,$$

etc., and collect terms having common denominators. For example, the terms independent of n (ignoring the common factor $[mp^2 (\lambda-\mu)^4]^{-1}$) sum to

$$\{p\lambda^2+(1-p)\mu^2 - [p\lambda+(1-p)\mu]^2\} \mu^2(\mu+2\lambda)^2$$

$$+ \{p\lambda^4+(1-p)\mu^4 - [p\lambda^2+(1-p)\mu^2]^2\} (\lambda+2\mu)^2$$

$$+ p\lambda^6+(1-p)\mu^6 - [p\lambda^3+(1-p)\mu^3]^2$$

$$- 2\{p\lambda^3+(1-p)\mu^3 - [p\lambda+(1-p)\mu][p\lambda^2+(1-p)\mu^2]\} (5\lambda\mu^2+2\mu^3+2\mu\lambda^2)$$

$$+ 2\{p\lambda^4+(1-p)\mu^4 - [p\lambda+(1-p)\mu][p\lambda^3+(1-p)\mu^3]\} (\mu^2+2\lambda\mu)$$

$$- 2\{p\lambda^5+(1-p)\mu^5 - [p\lambda^2+(1-p)\mu^2][p\lambda^3+(1-p)\mu^3]\} (\lambda+2\mu)$$

$$= 0 ;$$

of the terms that remain, those having $[mnp^2(\lambda-\mu)^4]^{-1}$ as a factor sum to $p\lambda(1-\lambda)(\lambda-\mu)^4$, etc.

After some computation, the elements of the asymptotic covariance matrix of λ^*, μ^*, p^* are in this way found to be

$$\sigma^2_{\lambda^*} = \frac{\lambda(1-\lambda)}{pmn} + \frac{8p\lambda^2(1-\lambda)^2+2(1-p)\mu^2(1-\mu)^2}{mn(n-1)p^2(\lambda-\mu)^2} + 6\,\frac{p\lambda^3(1-\lambda)^3+(1-p)\mu^3(1-\mu)^3}{mn(n-1)(n-2)p^2(\lambda-\mu)^4},$$

$$\sigma^2_{\mu^*} = \frac{\mu(1-\mu)}{(1-p)mn} + \frac{2p\lambda^2(1-\lambda)^2+8(1-p)\mu^2(1-\mu)^2}{mn(n-1)(1-p)^2(\lambda-\mu)^2} + 6\,\frac{p\lambda^3(1-\lambda)^3+(1-p)\mu^3(1-\mu)^3}{mn(n-1)(n-2)(1-p)^2(\lambda-\mu)^4},$$

$$\sigma^2_{p^*} = \frac{p(1-p)}{m} + 18\,\frac{p\lambda^2(1-\lambda)^2+(1-p)\mu^2(1-\mu)^2}{mn(n-1)(\lambda-\mu)^4} + 24\,\frac{p\lambda^3(1-\lambda)^3+(1-p)\mu^3(1-\mu)^3}{mn(n-1)(n-2)(\lambda-\mu)^6},$$

$$\sigma_{\lambda^*,\mu^*} = 4\,\frac{p\lambda^2(1-\lambda)^2+(1-p)\mu^2(1-\mu)^2}{mn(n-1)p(1-p)(\lambda-\mu)^2} + 6\,\frac{p\lambda^3(1-\lambda)^3+(1-p)\mu^3(1-\mu)^3}{mn(n-1)(n-2)p(1-p)(\lambda-\mu)^4}$$

$$\sigma_{\lambda^*,p^*} = 6\,\frac{2p\lambda^2(1-\lambda)^2+(1-p)\mu^2(1-\mu)^2}{mn(n-1)p(\mu-\lambda)^3} + 12\,\frac{p\lambda^3(1-\lambda)^3+(1-p)\mu^3(1-\mu)^3}{mn(n-1)(n-2)p(\mu-\lambda)^5}$$

$$\sigma_{\mu^*,p^*} = 6\,\frac{p\lambda^2(1-\lambda)^2+(1-p)\mu^2(1-\mu)^2}{mn(n-1)(1-p)(\mu-\lambda)^3} + 12\,\frac{p\lambda^3(1-\lambda)^3+(1-p)\mu^3(1-\mu)^3}{mn(n-1)(n-2)(1-p)(\mu-\lambda)^5}.$$

3.3 Asymptotic Relative Efficiency of the Moment Estimators

The asymptotic relative efficiency (ARE) of a consistent asymptotically normally distributed moment estimator $\hat{\theta}$ of a parameter θ relative to the corresponding maximum likelihood estimator, say θ^*, may be computed as (Cramér [10])

$$\text{ARE}\,(\hat{\theta}) = \frac{\sigma^2_{\theta^*}}{\sigma^2_{\hat{\theta}}} ,$$

where $\sigma^2_{\theta^*}$ is the Cramér-Rao lower bound and $\sigma^2_{\hat{\theta}}/m$ is the variance in the limiting distribution of $\hat{\theta}$. If $\hat{\theta} = (\hat{\theta}_1, \ldots \hat{\theta}_r)$ is an estimator of a vector parameter $\theta = (\theta_1, \ldots, \theta_r)$ with the components of $\hat{\theta}$ asymptotically jointly normally distributed with mean θ and covariance matrix $(\frac{1}{m}) \Sigma_{\hat{\theta}}$, the asymptotic relative efficiency may be computed in this way for each component of $\hat{\theta}$ or the components may be considered jointly. In this latter case, the joint asymptotic relative efficiency (JARE) of $\hat{\theta}$ relative to the maximum likelihood estimator θ^* is computed as the square of the ratio of the areas of the ellipses of concentration of the respective asymptotic normal distributions. (Cramér [10], Ch. 32) Since the areas of the ellipses of concentration are proportional to the determinants of the respective covariance matrices.

$$\text{JARE}\,(\hat{\theta}) = \frac{\det\left(\Sigma_{\theta^*}\right)}{\det\left(\Sigma_{\hat{\theta}}\right)} \tag{3.3.1}$$

The Cramér-Rao lower bound is determined by the relation

$$\Sigma^{-1}_{\theta^*} = E\left(\frac{\partial \log p(x)}{\partial \theta_i} \frac{\partial \log p(x)}{\partial \theta_j} \right) \tag{3.3.2}$$

where $P(\cdot)$ is given in equation (3.1.1). The elements of the inverse of $\Sigma_{\theta^*} = \Sigma_{\lambda^*,\mu^*,p^*}$ for the example under consideration are easily computed, e.g., the upper left-hand element of $\Sigma^{-1}_{\lambda^*,\mu^*,p^*}$ is

$$E\left(\frac{\partial \log p(x)}{\partial \lambda} \right)^2 = \sum_{x=0}^{\infty} \left[\frac{\partial p(x)}{\partial \lambda} \cdot \frac{1}{p(x)} \right]^2 p(x)$$

$$= \sum_{x=0}^{\infty} \left[\frac{p\, e^{-\lambda} \lambda^x (x-\lambda)}{x!\, \lambda} \right]^2 / \; p(x)$$

$$= \frac{p^2}{\lambda^2} \sum_{x=0}^{\infty} \left[\frac{e^{-\lambda} \lambda^x}{x!} \right]^2 (x-\lambda)^2 \,/\, p(x)$$

$$= \frac{p^2}{\lambda^2} \sum_{x=0}^{\infty} \frac{[P_1(x)(x-\lambda)]^2}{p(x)} \tag{3.3.3}$$

where $P_1(x) = \dfrac{\lambda^x e^{-\lambda}}{x!}$, and

$$E\left(\frac{\partial \log p(x)}{\partial \mu} \right)^2 = \sum_{x=0}^{\infty} \left[\frac{\partial p(x)}{\partial \mu} \cdot \frac{1}{p(x)} \right]^2 p(x)$$

$$= \sum_{x=0}^{\infty} \left[\frac{(1-p)\ e^{-\mu}\ \mu^{x}(x-\mu)}{x!\ \mu} \right]^2 \Big/\ p(x)$$

$$= \frac{(1-p)^2}{\mu^2} \sum_{x=0}^{\infty} \left[\frac{e^{-\mu}\ \mu^{x}}{x!} \right]^2 (x-\mu)^2 / p(x)$$

$$= \frac{(1-p)^2}{\mu^2} \sum_{x=0}^{\infty} \frac{[p_2(x)(x-\mu)]^2}{p(x)} \qquad (3.3.4)$$

where $p_2(x) = \dfrac{\mu^{x}\ e^{-\mu}}{x!}$.

Similarly,

$$E\left(\frac{\partial \log p(x)}{\partial p} \right)^2 = \sum_{x=0}^{\infty} \left[\frac{\partial p(x)}{\partial p} \cdot \frac{1}{p(x)} \right]^2 p(x)$$

$$= \sum \frac{[p_1(x) - p_2(x)]^2}{p(x)} \ . \qquad (3.3.5)$$

The desired matrix may be written

$$\sum_{\theta^*}^{-1} = \sum_{\lambda^*,\mu^*,p^*}^{-1} =$$

$$\begin{bmatrix} p^2 \sum \frac{[p_1(x)(x-\lambda)]^2}{\lambda^2 p(x)} & p(1-p) \sum \frac{p_1(x)p_2(x)(x-\lambda)(x-\mu)}{\lambda\mu p(x)} & p \sum \frac{p_1(x)(x-\lambda)[p_1(x)-p_2(x)]}{\lambda p(x)} \\ - & (1-p)^2 \sum \frac{[p_2(x)(x-\mu)]^2}{\mu^2 p(x)} & (1-p) \sum \frac{p_2(x)(x-\mu)[p_1(x)-p_2(x)]}{\mu p_2(x)} \\ - & - & \sum \frac{[p_1(x)-p_2(x)]^2}{p(x)} \end{bmatrix} \qquad (3.3.6)$$

Equation (3.3.6) is the Cramér-Rao lower bound covariance matrix for estimating the parameters of a mixture of two Poisson distributions.

3.4 Estimation of the Poisson Parameters When p (proportion) is known

We again assume the $\lambda < \mu$. Note that in Section (3.1) this assumption was not restrictive. In the present case, however, this can be a very real restriction. Unless $p = 1/2$, it is not sufficient when p is known to simply that $\lambda \neq \mu$; it must be known specifically that p is the proportion in which the population having smaller mean is present in the mixture (see Rider's comment [25] on this problem).

Under this assumption a development similar to that of Section (3.1) yields as functions of the first two factorial moments the moment estimators

$$\left.\begin{aligned} \lambda^{**} &= F_1 - \left[\frac{1-p}{p}\,(F_2-F_1^2)\right]^{1/2} \\ \mu^{**} &= F_1 + \left[\frac{p}{1-p}\,(F_2-F_1^2)\right]^{1/2} . \end{aligned}\right\} \qquad (3.4.1)$$

These estimators are efficient for $n = 2$. We shall now show, however, that the asymptotic relative efficiencies of the estimators of equation (3.4.1) tend to 0 rather than to 1 as $n \to \infty$.

The entries of the covariance matrix $\frac{1}{m} \Sigma_{\lambda^{**},\, \mu^{**}}$ are computed as before. We get

$$\left[\frac{\partial\lambda^{**}}{\partial F_1}\bigg|_{f_1} \quad \frac{\partial\mu^{**}}{\partial F_2}\bigg|_{f_2} \right] = \frac{1}{(\mu-\lambda)} \begin{bmatrix} \frac{1}{p} & -\frac{1}{2p} \\ -\frac{\lambda}{1-p} & \frac{1}{2(1-p)} \end{bmatrix}$$

So that by Cramér's theorem,

$$\sigma^2_{\lambda^{**}} = \frac{\lambda(1-\lambda)}{pmn} + \frac{1-p}{p}\,\frac{(\lambda-\mu)^2}{4m} + \frac{p\lambda^2(1-\lambda)^2+(1-p)\mu^2(1-\mu)^2}{2mn(n-1)p^2(\lambda-\mu)^2}$$

$$\sigma^2_{\mu^{**}} = \frac{\mu(1-\mu)}{(1-p)mn} + \frac{p}{1-p}\,\frac{(\lambda-\mu)^2}{4m} + \frac{p\lambda^2(1-\lambda)^2+(1-p)\mu^2(1-\mu)^2}{2mn(n-1)(1-p)^2(\lambda-\mu)^2} \tag{3.4.2}$$

$$\sigma_{\lambda^{**},\,\mu^{**}} = \frac{(\lambda-\mu)^2}{4m} - \frac{p\lambda^2(1-\lambda)^2+(1-p)\mu^2(1-\mu)^2}{2m(n-1)(1-p)p(\lambda-\mu)^2} \quad .$$

In this case, the Cramér-Rao lower bound covariance matrix for estimating the parameters of a mixed-Poisson distribution is

$$\begin{bmatrix} \frac{p^2}{\lambda^2} \sum \frac{[p_1(x)(x-\lambda)]^2}{p(x)} & \frac{p(1-p)}{\lambda\mu} \sum \frac{[p_1(x)p_2(x)(x-\lambda)(x-\mu)}{p(x)} \\ - & \frac{(1-p)^2}{\mu^2} \sum \frac{[p_2(x)(x-\mu)]^2}{p(x)} \end{bmatrix}$$

3.5 Mixtures of Three or More Poisson Distributions

Introductory Remarks

We shall now consider the estimation problem for a mixture of an arbitrary number, r , of Poisson distributions. Let $X_1,\dots,X_m$ be independent and identically distributed random variables, each having the distribution (Blischke [4])

$$P(X_i = x_i) = \sum_{j=1}^{r} p_j P_j(x_i) \tag{3.5.1}$$

where

$$P_j(x_i) = \frac{\lambda_j^{x_i} e^{-\lambda_j}}{x_i!}$$

with $0 < P_i < 1$ for $i = 1,2,\dots,r$, $\sum_{i=1}^{r} P_i = 1$, $n \geq 2r-1$, and all parameters (except n) unknown. Equation (3.5.1) is a mixture of r Poisson distributions. In this case there are $(2r-1)$ parameters to estimate. Estimators for these $(2r-1)$ parameters can be constructed by the methods of section (3.1) as functions of the sample factorial moments $F_1,F_2,\dots,F_{2r-1}$, defined in equation (3.1.2). In this case the expectation of the k^{th} sample factorial moment becomes

$$f_k = \sum_{i=1}^{r} p_i \lambda_i^k \; . \tag{3.5.2}$$

The verification of equation (3.5.2) is identical with that of equation (3.1.3). Thus the moment estimators are constructed by solving $f_1, f_2, \ldots, f_{2r-1}$ for $\lambda_1, \ldots, \lambda_r$, $p_1, \ldots, p_{r-1}$ and substituting F_k's for f_k's in the solution. We shall show that $r > 4$, this operation will result in a polynomial in one of the parameters of degree five or more. Such a polynomial cannot be solved algebrically, so that the estimators cannot be written explicitely but only as roots of certain polynomials.

Construction of the Moment Estimators

To illustrate further the method of attack required in computing estimators for mixtures of Poissons, we shall now outline the construction f or the case $r = 3$. (In this case the estimators can be obtained explicitely and the existance of a unique solution of $f_1, \ldots, f_5$ for $\lambda_1, \lambda_2, \lambda_3$, p_1, and p_2 is a proof of identifiability for $n \geq 2r-1 = 5$.)

First solve f_1 and f_2 for p_1 and p_2 to obtain

$$p_1 = \frac{f_2 - \lambda_2 f_1 - \lambda_3 f_1 + \lambda_2 \lambda_3}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} \qquad (3.5.3)$$

$$p_2 = \frac{f_2 - \lambda_1 f_1 - \lambda_3 f_1 + \lambda_1 \lambda_3}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} \quad . \qquad (3.5.4)$$

Substitution of equations (3.5.3) and (3.5.4) into f_3 yields

$$f_3 = p_1(\lambda_1^3-\lambda_3^3) + p_2(\lambda_2^3-\lambda_3^3) + \lambda_3^3$$

$$= \frac{1}{\lambda_1-\lambda_2} [(f_2-\lambda_2 f_1-\lambda_3 f_1+\lambda_2\lambda_3)(\lambda_1^2+\lambda_1\lambda_3+\lambda_3^2)$$

$$- (f_2-\lambda_1 f_1-\lambda_3 f_1+\lambda_1\lambda_3)(\lambda_2^2+\lambda_2\lambda_3+\lambda_3^2) + \lambda_3^3$$

$$= (f_2-\lambda_3 f_1)(\lambda_1+\lambda_2+\lambda_3) - \lambda_1\lambda_2(f_1-\lambda_3)$$

$$+ \lambda_3^2(f_1-\lambda_3) + \lambda_3^3$$

$$= (\lambda_1+\lambda_2+\lambda_3)f_2 - (\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3)f_1$$

$$+ \lambda_1\lambda_2\lambda_3 \tag{3.5.5}$$

Similarly,

$$f_4 = p_1(\lambda_1^4-\lambda_3^4) + p_2(\lambda_2^4-\lambda_3^4) + \lambda_3^4$$

$$= \frac{1}{\lambda_1-\lambda_2} [(f_2-\lambda_2 f_1-\lambda_3 f_1+\lambda_2\lambda_3)(\lambda_1^3+\lambda_1^2\lambda_3+\lambda_1\lambda_3^2+\lambda_3^2)$$

$$+ (f_2-\lambda_1 f_1-\lambda_3 f_1+\lambda_1\lambda_3)(\lambda_2^3+\lambda_2^2\lambda_3+\lambda_2\lambda_3^2+\lambda_3^3) + \lambda_3^4$$

$$= (\lambda_1+\lambda_2+\lambda_3)f_3 - (\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3)f_2 + \lambda_1\lambda_2\lambda_3 f_1 \tag{3.5.6}$$

From equation (3.5.5)

$$\lambda_1 = \frac{f_3-\lambda_2 f_2-\lambda_3 f_2+\lambda_2\lambda_3 f_1}{f_2-\lambda_2 f_1-\lambda_3 f_1+\lambda_2\lambda_3} \quad .$$

Substituting this into equation (3.5.6) to obtain

$$(f_4-\lambda_2f_3-\lambda_3f_3+\lambda_2\lambda_3f_2)(f_2-\lambda_2f_1-\lambda_3f_1+\lambda_2\lambda_3) =$$

$$= (f_3-\lambda_2f_2-\lambda_3f_2+\lambda_2\lambda_3f_1)^2$$

or

$$0 = (f_1f_3-\lambda_3f_3+\lambda_3^2f_2-f_2^2-\lambda_3^2f_1^2+\lambda_3f_1f_2)\lambda_2^2$$

$$+ (\lambda_3f_4-f_1f_4+f_2f_3-\lambda_3^2f_3-\lambda_3f_2^2+\lambda_3^2f_1f_2)\lambda_2$$

$$+ (f_2f_4-\lambda_3f_1f_4+\lambda_3f_2f_3+\lambda_3^2f_1f_3-f_3^2-\lambda_3^2f_2^2)$$

$$\equiv Q_1(\lambda_3)\cdot\lambda_2^2 + Q_2(\lambda_3)\cdot\lambda_2 + Q_3(\lambda_3)\ , \qquad (3.5.7)$$

say. As in the solution for two Poissons, elimination of λ_2 from equations (3.5.4) and (3.5.5) will result in equation (3.5.7) with λ_1 replacing λ_2 . Thus the two roots of the equation

$$Q_1(\lambda_3)x^2 + Q_2(\lambda_3)x + Q_3(\lambda_3) = 0$$

are λ_1 and λ_2 expressed as functions of λ_3 and the factorial moments involved. In particular,

$$\lambda_1 = \frac{-\,Q_2 - (Q_2^2 - 4Q_1Q_3)^{1/2}}{2Q_1} \qquad (3.5.8)$$

$$\lambda_2 = \frac{-\,Q_2 + (Q_2^2 - 4Q_1Q_3)^{1/2}}{2Q_1} \qquad (3.5.9)$$

Finally, substitute equations (3.5.3) and (3.5.4), then (3.5.8) and (3.5.9) into f_5 . As above,

$$f_5 = p_1(\lambda_1^5-\lambda_3^5) + p_2(\lambda_2^5-\lambda_3^5) + \lambda_3^5$$

$$= (\lambda_1+\lambda_2+\lambda_3)f_4 - (\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3)f_3 +$$

$$+ \lambda_1\lambda_2\lambda_3 f_2$$

$$= (\lambda_1+\lambda_2)(f_4-\lambda_3 f_3) - \lambda_1\lambda_2(f_3-\lambda_3 f_2) + \lambda_3 f_4 \quad .$$

From equations (3.5.8) and (3.5.9) ,

$$\lambda_1+\lambda_2 = - Q_2/Q_1$$

and

$$\lambda_1\lambda_2 = Q_3/Q_1 \quad .$$

Thus

$$0 = Q_1(\lambda_3) \cdot (f_5-\lambda_3 f_4) + Q_2(\lambda_3) \cdot (f_4-\lambda_3 f_3) +$$

$$+ Q_3(\lambda_3) \cdot (f_3-\lambda_3 f_2)$$

$$= (f_1 f_3-\lambda_3 f_3+\lambda_3^2 f_2-f_2^2-\lambda_3^2 f_1^2+\lambda_3 f_1 f_2)(f_3-\lambda_3 f_4)$$

$$- (f_1 f_4-\lambda_3 f_4+\lambda_3^2 f_3-f_2 f_3-\lambda_3^2 f_1 f_2+\lambda_3 f_2^2)(f_4-\lambda_3 f_2)$$

$$+ (f_2 f_4-\lambda_3 f_1 f_4+\lambda_3 f_2 f_3+\lambda_3^2 f_1 f_3-\lambda_3^2-\lambda_3^2 f_2^2)(f_3-\lambda_3 f_2) \quad .$$

or

$$
\begin{aligned}
&(f_2f_4+2f_1f_2f_3-f_3^2-f_2^3-f_1^2f_4)\lambda_3^3 \\
&\quad - (f_2f_5+f_1f_2f_4+f_1f_3^2-f_3f_4-f_2^2f_3-f_1^2f_5)\lambda_3^2 \\
&\quad + (f_1f_3f_4+f_2^2f_4+f_3f_5-f_1f_2f_5-f_4^2-f_2f_3^2)\lambda_3 \\
&\quad - (f_1f_3f_5+2f_2f_3f_4-f_2^2f_5-f_1f_4^2-f_3^3) = 0 \quad . \qquad (3.5.10)
\end{aligned}
$$

Again by symmetry, solution for λ_1 or λ_2 instead of λ_3 will result in the cubic equation (3.5.10) with λ_3 replaced by λ_1 or λ_2 . Thus the three roots of equation (3.5.10) are λ_1, λ_2 and λ_3 .

CHAPTER IV

MIXED TRUNCATED POISSON DISTRIBUTIONS WITH MISSING ZERO CLASSES

4.0 Introduction

The first three sample moments were employed by Rider [26] in obtaining explicit estimators for parameters of mixtures of two Poisson distributions. In 1963, Rider's results were extended by Cohen [9]. The simplification is achieved through use of factorial rather than ordinary moments. The extensions involve consideration of alternate estimators for the case considered by Rider [26], using the first two sample moments plus the zero sample frequency; mixtures of truncated Poisson distributions; and some factors involved in deciding when sample data are indeed from a mixed rather than from a non-mixed distribution (Cohen [9]).

4.1 Estimates Based on First Two Sample Moments and Zero Sample Frequency

Let λ and μ be the parameters of the mixed-Poisson distribution that have been mixed in the proportions p and $(1-p)$ respectively. The probability function of the mixed distribution is then

$$g(x) = p\,\frac{\lambda^x e^{-\lambda}}{x!} + (1-p)\,\frac{\mu^x e^{-\mu}}{x!}\,, \qquad x = 0,1,2,\ldots \qquad (4.1.1)$$

where without any loss of generality, we let $\lambda > \mu$. The k^{th} factorial moment of x can be written as

$$m_{[k]} = p\lambda^k + (1-p)\mu^k \quad .$$

The estimating equations are

$$\bar{x} - \mu = p(\lambda-\mu)$$

$$\nu_{[2]} - \mu^2 = p(\lambda^2-\mu^2)$$

$$\frac{n_o}{n} - e^{-\mu} = p[e^{-\lambda} - e^{-\mu}] \tag{4.1.2}$$

where

$$\nu_{[k]} = \sum_{x=o}^{R} x(x-1)\ldots(x-k+1)\,\frac{n_x}{n} \tag{4.1.3}$$

in which R is the largest observed value of x , n_x is the sample frequency of x , n_o is the number zero observations in the sample and n is the total sample size; that is

$$\left(n = \sum_{x=o}^{R} n_x \right) \quad .$$

On eliminating p between the first two, and between the first and third equations of (4.1.2), we have

$$\lambda = \left[\nu_{[2]} - \bar{x}\,\mu \right] / (x-\mu) = G(\mu)$$

$$(\bar{x}-\mu)/(\lambda-\mu) = \left[\frac{n_o}{n} - e^{-\mu} \right] / \left[e^{-\lambda} - e^{-\mu} \right] \qquad (4.1.4)$$

When $\lambda = G(\mu)$ is substituted into the second equation of (4.1.4), we obtain

$$[\bar{x}-\mu]/[G(\mu)-\mu] = \left[\frac{n_o}{n} - e^{-\mu} \right] / \left[e^{-G(\mu)} - e^{-\mu} \right] \qquad (4.1.5)$$

which can be solved for μ^* using standard iterative procedures. With μ^* determined from (4.1.5), λ^* follows from the first equation of (4.1.4) as

$$\lambda^* = \left[\nu_{[2]} - \bar{x}\,\mu^* \right] / (\bar{x}-\mu^*) \qquad (4.1.6)$$

and p^* follows from the first equation of (4.1.2) as

$$p^* = (\bar{x}-\mu^*)/(\lambda^*-\mu^*) \quad . \qquad (4.1.7)$$

4.2 Mixed Truncated Poisson Distributions with Missing Zero Classes

In this case, the probability function of the mixed distribution is written as

$$f(x) = p\,\frac{\lambda^x e^{-\lambda}}{x!\,(1-e^{-\lambda})} + (1-p)\,\frac{\mu^x e^{-\mu}}{x!\,(1-e^{-\mu})} \;, \quad x = 1,2,\ldots \qquad (4.2.1)$$

The k^{th} factorial moment is

$$m_{[k]} = p \frac{\lambda^k}{1-e^{-\lambda}} + (1-p) \frac{\mu^k}{1-e^{-\mu}}$$

$$= p \frac{\lambda^k}{1-e^{-\lambda}} + \frac{\mu^k}{1-e^{-\mu}} - p \frac{\mu^k}{1-e^{-\mu}}$$

$$= p \left[\frac{\lambda^k}{1-e^{-\lambda}} - \frac{\mu^k}{1-e^{-\mu}} \right] + \frac{\mu^k}{1-e^{-\mu}}$$

The estimating equations resulting from equating sample moments to corresponding distribution moments (4.2.2), in this case, are

$$\left[\bar{x} - \frac{\mu}{1-e^{-\mu}} \right] = p \left[\frac{\lambda}{1-e^{-\lambda}} - \frac{\mu}{1-e^{-\mu}} \right] ,$$

$$\left[\nu_{[2]} - \frac{\mu^2}{1-e^{-\mu}} \right] = p \left[\frac{\lambda^2}{1-e^{-\lambda}} - \frac{\mu^2}{1-e^{-\mu}} \right]$$

$$\left[\nu_{[3]} - \frac{\mu^3}{1-e^{-\mu}} \right] = p \left[\frac{\lambda^3}{1-e^{-\lambda}} - \frac{\mu^3}{1-e^{-\mu}} \right] \tag{4.2.3}$$

If the first of the above equations is multiplied by μ and subtracted from the second, and then the second is multiplied by μ and subtracted from the third, the following equations are obtained.

$$\left.\begin{aligned} \nu_{[2]} - \mu \bar{x} &= p\lambda(\lambda-\mu)/(1-e^{-\lambda}) , \\ \nu_{[3]} - \mu\nu_{[2]} &= p\lambda^2(\lambda-\mu)/(1-e^{-\lambda}) \end{aligned}\right\} \tag{4.2.4}$$

When the second equation of (4.2.4) is divided by the first, the result is

$$\lambda = \frac{\nu_{[3]} - \mu\nu_{[2]}}{\nu_{[2]} - \mu\bar{x}} = H(\mu) \tag{4.2.5}$$

Then elimination of p between the first two equations of (4.2.3), yields

$$\frac{\left[\bar{x} - \frac{\mu}{1-e^{-\mu}}\right]}{\left[\frac{\lambda}{1-e^{-\lambda}} - \frac{\mu}{1-e^{-\mu}}\right]} = \frac{\left[\nu_{[2]} - \frac{\mu^2}{1-e^{-\mu}}\right]}{\left[\frac{\lambda^2}{1-e^{-\lambda}} - \frac{\mu^2}{1-e^{-\mu}}\right]} \tag{4.2.6}$$

Finally, when λ in (4.2.6) is replaced by $H(\mu)$, the result is the following equation in μ only.

$$\frac{\left[\bar{x} - \frac{\mu}{1-e^{-\mu}}\right]}{\left[\frac{H(\mu)}{1-e^{-H(\mu)}} - \frac{\mu}{1-e^{-\mu}}\right]} = \frac{\left[\nu_{[2]} - \frac{\mu^2}{1-e^{-\mu}}\right]}{\left[\frac{H^2(\mu)}{1-e^{-H(\mu)}} - \frac{\mu^2}{1-e^{-\mu}}\right]} \tag{4.2.7}$$

Equation (4.2.7) can be solved for μ^* only using standard iterative procedures. With μ^* determined from (4.2.7), λ^* may be obtained from (4.2.5) as

$$\lambda^* = \frac{\nu_{[3]} - \mu^*\nu_{[2]}}{\nu_{[2]} - \mu^*\bar{x}} , \tag{4.2.8}$$

and p^* may be obtained from the first equation of (4.2.4) as

$$p^* = \left[\frac{\bar{x} - \lambda^*}{1-e^{-\mu^*}} \right] \Big/ \left[\frac{\lambda^*}{1-e^{-\lambda^*}} - \frac{\mu^*}{1-e^{-\mu^*}} \right] \tag{4.2.9}$$

4.3 Which Distribution is Applicable

Whenever it is known in advance of selecting a sample that one of the mixed distributions discussed in the preceeding section of this chapter is applicable, then it is relatively simple task to calculate estimates using the appropriate estimating equations. In the more general situations, it is necessary to determine from the sample data which distribution most adequately accounts for the observations at hand. The latter problem is more difficult to deal with and much remains to be learned concerning its solution.

In this section, we consider the following illustrative example employed by Rider [26] .

TABLE II

x	n_x
0	830
1	638
2	327
3	137
4	49
5	15
6	3
7	1

After a casual examination of these data, one might be led to suggest that they originated with an ordinary Poisson distribution whereas in fact they represent a mixture of equal proportions of two Poisson distributions with

$$\lambda = 1.5 \quad \text{and} \quad \mu = 0.5$$

respectively. An indication of the extent to which a simple Poisson fits these or similar data might be obtained by calculating the set of estimates

$$\lambda^{**} = (x+1)\,\frac{n_{x+1}}{n_x} \tag{4.3.1}$$

and the set of estimates λ^* for which

$$\frac{e^{-\lambda^{x}}\lambda^{**x}}{x!} = \frac{n_x}{n} \qquad (4.3.2)$$

The calculation of the λ^{**} presents no difficulties, and λ^{*} can be read directly from the tables of the Poisson function. These estimates for Rider's example are given in the table below.

Sample moments for these data are

$$\bar{x} = 0.9995 \; ; \quad \nu_{[2]} = 1.248 \; ; \quad \nu_{[3]} = 1.734$$

TABLE III

DISTRIBUTION OF SURFACE DEFECTS IN 2000 PIECES OF ENAMELED WARE

x	n_x	n_x/n	λ^{*}	λ^{**}
0	830	0.4150	0.879	0.769
1	638	0.3190	0.556	1.025
2	327	0.1635	0.894	1.257
3	137	0.0685	1.06	1.431
4	49	0.0245	1.17	1.531
5	15	0.0075	1.26	1.200
6	3	0.0015	1.24	2.333
7	1	0.0005	1.39	
TOTAL	2000	-	-	-

The lack of consistency among the various estimates λ^* and λ^{**} clearly suggests that these data cannot be satisfactorily fitted by the ordinary Poisson distribution. The mixed-Poisson distribution provides an almost perfect fit for these data, and estimates computed by substituting applicable sample values into equations (2.7.1) and (2.5.5) are

$$\lambda^* = 1.4767$$

$$\mu^* = 0.4777$$

and

$$p^* = 0.5224 \ .$$

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